Lecture 30

Determinants, Linear transformations, and Volume

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$[T] = A$

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

- $\det A = \pm$ area of parallelogram P with vertices $\vec{0}, \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\substack{\vec{a}_1 \\ = \\ T(\vec{e}_1)}}, \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\substack{\vec{a}_2 \\ = \\ T(\vec{e}_2)}}, \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{a}_1 + \vec{a}_2}$

- $P = T$ (unit square)

ALT

Area $P = (\pm \det A)$ Area(unit square)

Area[T(unit square)] $= |\det A| \cdot$ Area(unit square)

- **Thm**: Given $A \in M_{n\times n}$, let $T: \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation with matrix $[T] = A$. Then $\forall$ region $\Omega \subset \mathbb{R}^n$ we have:

$$\mathrm{vol}(T(\Omega)) = |\det A| \cdot \mathrm{Vol}(\Omega)$$

$\boxed{\text{Ex}}$ If D = unit disk in $\mathbb{R}^2$ then by theorem Area $T(D) = |\det A| \cdot \text{Area}(D) = 2\pi$

Why is Thm true?

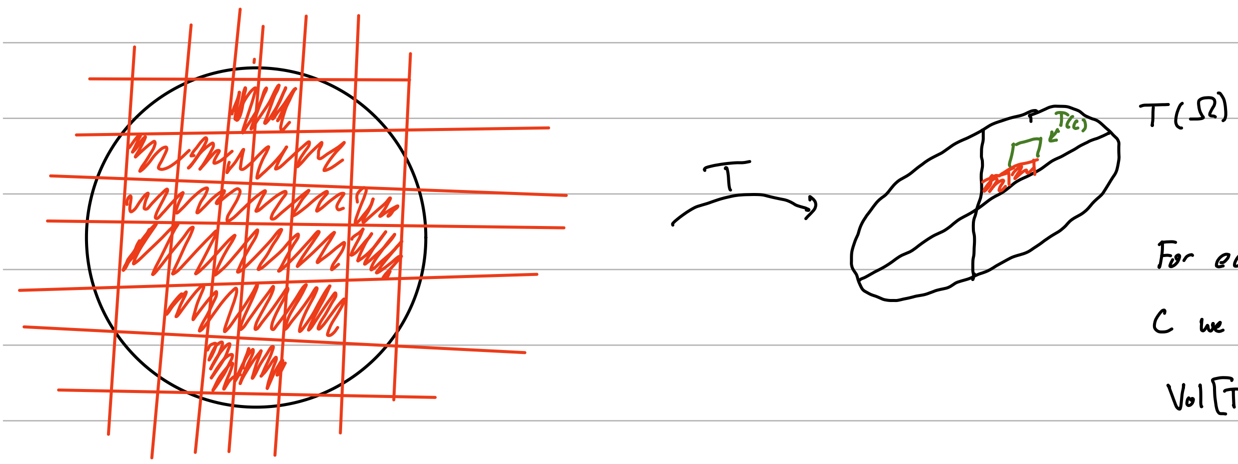
(1) it works if $\Omega = P(\vec{e}_1, \dots, \vec{e}_n)$ = unit cube in $\mathbb{R}^2$

i.e. $\text{Vol } T(\Omega) = \text{Vol}\Big[P\big[T(\vec{e}_1), \dots, T(\vec{e}_n)\big]\Big]$

$= \text{Vol } P(\vec{a}_1, \dots, \vec{a}_n)$

$= |\det A| \cdot \text{Vol}(\Omega)$

• To go from $\Omega = P(\vec{e}_1, \dots, \vec{e}_n)$ to general Ω, approximate Ω from inside by a union of non-overlapping cubes



For each little cube C we have:

$\text{Vol}[T(C)] = |\det A| \text{ Vol}(C)$

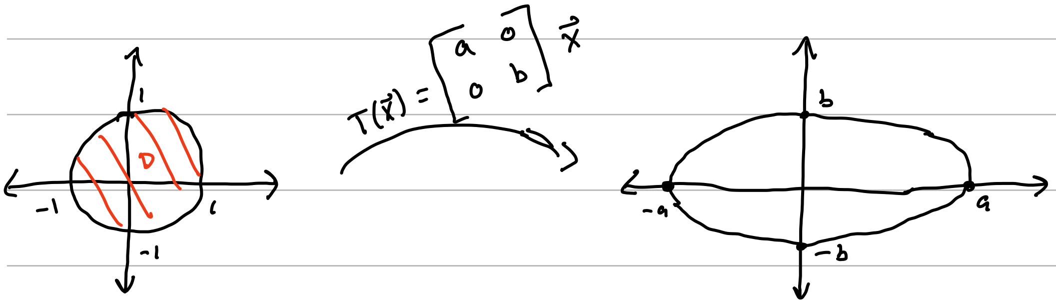
• Hence:

$$\text{Vol}[T(\Omega)] \approx \sum_{\substack{c = \text{little} \\ \text{cubes in } \Omega}} \text{Vol}[T(c)] = \sum |\det A| \text{ Vol}(c) \approx |\det A| \cdot \text{Vol}(\Omega)$$

$\blacksquare$

• Remember: $\det(\epsilon A) = \epsilon^n (\det A)$, where ϵ is a constant

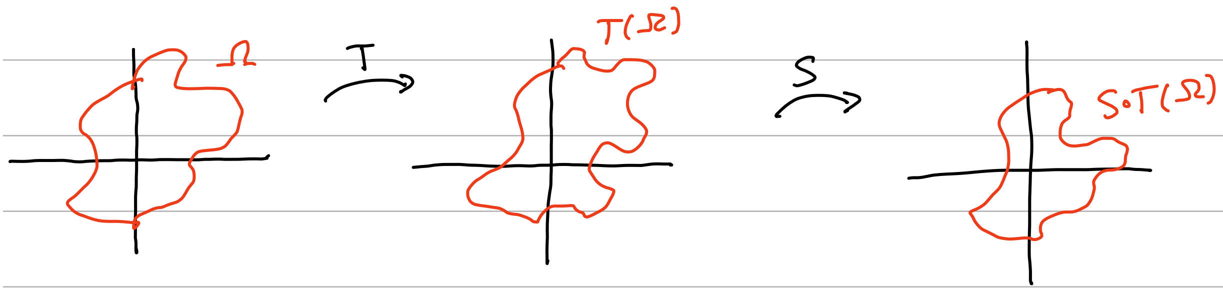
$\boxed{\text{Ex}}$ What is area of ellipse: E



$$\text{Vol } E = \text{Vol } T(D) = \left| \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} \right| \text{Vol}(D)$$

$$= ab \ \text{Vol}(D) = \pi a b$$

• $T: \mathbb{R}^n \to \mathbb{R}^n \quad [T] = A \quad , \quad S: \mathbb{R}^n \to \mathbb{R}^n \quad [S] = B$



• OTOH

$$\text{Vol}\left[S \circ T(\Omega)\right] = |\det(BA)| \ \text{Vol}(\Omega)$$

$$\|$$

$$|\det B| \ \text{Vol}\left[T(\Omega)\right] = |\det(BA)| \ \text{Vol}(\Omega)$$

$$\|$$

$$|\det B| \, |\det A| \ \text{Vol}(\Omega) = |\det(BA)| \ \text{Vol}(\Omega)$$

☆ "Cor": $\forall$ matrices $A, B \in M_{n \times n}$ we have $\det(BA) = \det(B)[\det(A)]$

↳ Warning: This cor does not work for sums of matrices

$\boxed{\text{E.g.}}$ $\det\left[\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 3 & 0 \end{pmatrix}\right] \neq \det\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} + \det\begin{pmatrix} 1 & -1 \\ 3 & 0 \end{pmatrix}$

$\boxed{\text{Ex}}$ $\det\left(\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}^{99}\right)$

$$= \det\left[\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}\right]$$

$$\overset{\text{"Cor"}}{=} \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} \cdot \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} \cdot \ldots \cdot \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = \left(\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}\right)^{99} = (8 \cdot 3)^{99} = \boxed{(5)^{99}}$$

☆ • Cor : If $A \in M_{n \times n}$ is invertible, then $\det(A^{-1}) = \dfrac{1}{\det A}$

Proof : He proved on board

☆ • Cor : If $A \in M_{n \times n}$ is orthogonal then $\det A = \pm 1$

Proof: A orthogonal $\rightarrow AA^T = I$

$$\rightarrow \det(AA^T) = 1$$

$$\rightarrow [\det(A)][\det(A^T)] = 1$$

$$\rightarrow (\det A)^2 = 1 \quad \blacksquare$$

* rotations give $\det = 1$, reflections give you $\det = -1$

☆ • Cor : If $A, B \in M_{n \times n}$ are similar, then $\det A = \det B$

Proof : A similar to $B \Leftrightarrow A = P^{-1}BP$ ← check this formula

$$\rightarrow \det A = \det(P^{-1}BP) = (\det P^{-1})(\det B)(\det P)$$

$$= \det B$$

Recall: If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and $\beta \subset \mathbb{R}^n$ is a basis,

then $[T]_\beta \neq [T]_\varepsilon$

but the matrices are similar so $\det(T)_\beta = \det[T]$